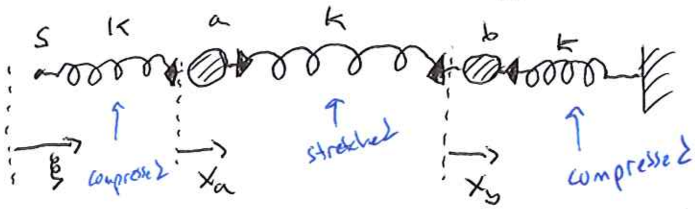
Classical & Quantum Waves

Forced oscillations of coupled oscillators

- Recall that for single harmonic oscillators, oscillation amplitude becomes very large when periodic driving force is applied at its natural frequency of oscillation. Small amplitude for other frequencie.
- What happens for coupled oscillator case when there are more than one natural frequencies, pertaing to differat normal modes?

  → One would guess that amplitude becomes large when driven close to either normal frequency → This is indeed true.

- Consider the following system:



$\xi = a\cos\omega t$

(at this snapshot in time)

Eq. of motions for mass a:

$$m\frac{d^2x_a}{dt^2} = -\underbrace{k(x_a-\xi)}_{} + \underbrace{k(x_b-x_a)}_{}$$

left spring — when >0, left spring is stretched, providing restoring force back to EP (so minus sign)

middle spring — when >0, middle spring is stretched, which pulls mass a away from EP (so + sign)

$$\Rightarrow \frac{d^2x_a}{dt^2} + \frac{2k}{m}x_a - \frac{k}{m}x_b = \frac{F_0}{m}\cos\omega t$$

where $F_0 = ka$

Eq. motion mass b:

$$m\frac{d^2x_b}{dt^2} = -kx_b - \underbrace{k(x_b-x_a)}_{}$$

right spring. $-$ b/c $x_b>0$ forces mass b to EP

middle spring. $-$ b/c $(x_b-x_a)>0$ forces mass b to EP

$$\Rightarrow \frac{d^2x_b}{dt^2} - \frac{k}{m}x_a + \frac{2k}{m}x_b = 0$$

- Adding and subtracting the two eqs. of motion (similar to last lecture ~~[illegible]~~ for coupled oscillators)

- $$\frac{d^2(x_a+x_b)}{dt^2} + \frac{k}{m}(x_a+x_b) = \frac{F_0}{m}\cos\omega t$$

- $$\frac{d^2(x_a-x_b)}{dt^2} + \frac{3k}{m}(x_a-x_b) = \frac{F_0}{m}\cos\omega t$$

$\rightarrow$ change variables to $q_1 = (x_a + x_b)$, $q_2 = (x_a - x_b)$

$$\Rightarrow \left.\begin{aligned} \frac{d^2 q_1}{dt^2} + \frac{k}{m} q_1 &= \frac{F_0}{m}\cos\omega t \\ \frac{d^2 q_2}{dt^2} + \frac{3k}{m} q_2 &= \frac{F_0}{m}\cos\omega t \end{aligned}\right\}$$

For each indep. coordinate q_1, q_2 get decoupled forced harmonic oscillators

$\rightarrow$ Means we can use solutions from ~~FHO~~ Forced HO

$\rightarrow$ $q_1 = C_1 \cos\omega t$ $\qquad q_2 = C_2 \cos\omega t$

$$C_1 = \frac{F_0/m}{(\omega_1^2 - \omega^2)} \qquad C_2 = \frac{F_0/m}{(\omega_2^2 - \omega^2)}$$

$$\omega_1^2 = \frac{k}{m} \qquad \omega_2^2 = \frac{3k}{m}$$

[refer back to Lecture 7 notes, or King eqs. 3.5a, 3.7a]

- $C_1, C_2 \rightarrow \infty$ when $\omega \rightarrow \omega_1, \omega_2$ (b/c no damping, as before)

→ Setup has two resonance freqs. (where amplitude large) ~~corresponding to~~ corresponding to the two normal freqs.

- We can express motion of masses a, b using:

$$x_a = \frac{1}{2}(q_1 + q_2) = \frac{1}{2}(C_1 + C_2)\cos\omega t$$

$$x_b = \frac{1}{2}(q_1 - q_2) = \frac{1}{2}(C_1 - C_2)\cos\omega t$$

- From these, we see that when $\omega \to \omega_1$, $|C_1| >> |C_2|$

→ $x_a \approx x_b$ i.e., masses oscillate <u>in phase</u>

When $\omega \to \omega_2$, $|C_2| >> |C_1|$, $x_a \approx -x_b$ → oscillate <u>out-of-phase</u>

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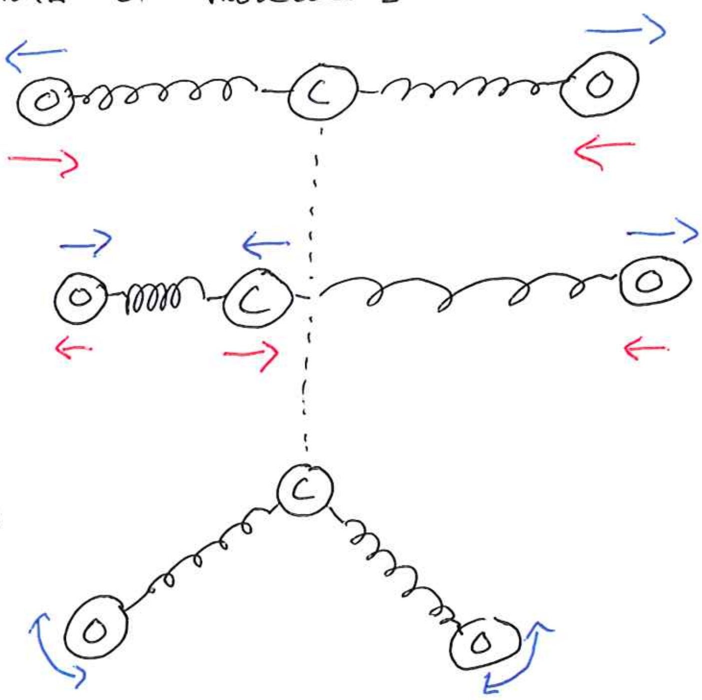
- Most coupled oscillator systems are too complicated to calculate the normal frequencies, but they can be ~~determined~~ exp. measured via resonance!

<u>Example:</u> Vibrations of molecules

- CO_2

(springs represent molecular bonds)

- Measured via absorption spectroscopy
  → shine light, see what frequencies are absorbed (power resonance curve)



<u>symm. stretch mode</u>

$\nu_s = 4 \times 10^{13}\,Hz$

$(\lambda_s \sim 7.5\,\mu m)$

<u>asymm. stretch mode</u>

$\nu_a = 7 \times 10^{13}\,Hz$

$(\lambda_a \sim 4.3\,\mu m)$

<u>bending</u> <u>mode</u>

$\nu_b = 2 \times 10^{13}\,Hz$

$(\lambda_b = 15\,\mu m)$

[google "CO_2 absorption spectrum"]

· CO_2 + global heating

→ Earth is at ~300K. This radiates light (via blackbody radiation) in the mid-infrared (1~10 μm) where CO_2 molecules absorb light due to their resonance frequencies.

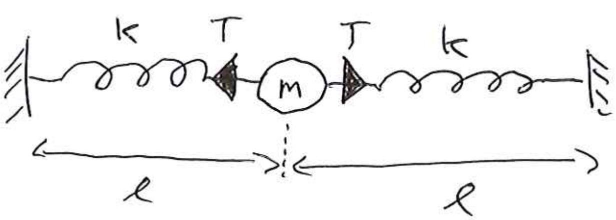
→ This traps heat within atmosphere, instead of letting it pass through atmosphere into outer space. This causes the earth to heat up. Now ya know why!

// Transverse Oscillations

· Up until this point, motion of masses coupled by springs always along line connecting the masses → "longitudinal oscillation/mode"

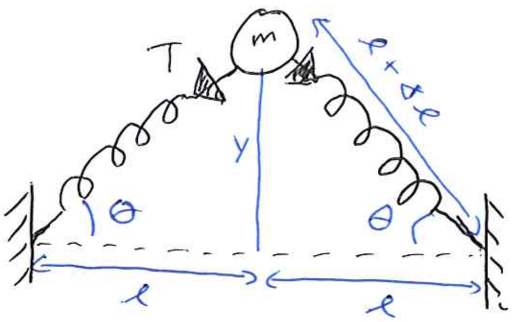
· Can also have oscillations ⊥ to line → "transverse oscillation/mode"

· Consider the following:



· Springs slightly stretched in equilibrium such that there is a Tension Force T exerted on mass in both directions (ie, $l >$ unstretched spring length)

· Displace mass vertically:



· For small angles Θ (much smaller than what I've drawn here!)

$$\ell \simeq \ell + \Delta\ell$$

→ So we can assume tension T is approx. constant during transverse oscillations

~~(b/c T = -ks~~

· Springs exert restoring force on mass ~~$F = -2T\sin\Theta$~~ along y-direction

$$\sin\Theta = \frac{-T_y}{T}$$

← y-component of tension

tension

$$\longrightarrow \quad T_y = -T\sin\Theta \quad \underline{\text{per spring}}$$

$\ominus$ b/c restoring back to EP

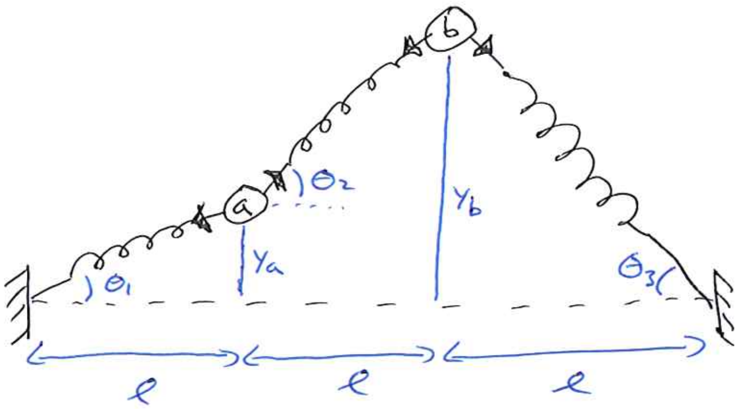
· Eq. of motion: $m\dfrac{d^2y}{dt^2} = -2T\sin\Theta \simeq -2T\dfrac{y}{\ell+\Delta\ell} \simeq -2T\dfrac{y}{\ell}$

$\Delta\ell$ small

$$\Rightarrow \frac{d^2y}{dt^2} = -\frac{2T}{m\ell}y \quad \Rightarrow \quad \omega_0^2 = \frac{2T}{m\ell} \quad \text{natural freq.}$$

→ One normal mode of vibration (b/c only one mass)

· Extend discussion to two masses connected by 3 springs:



(at some snapshot in time)

· Eq. of motion: (mass a)

$$m\frac{d^2y_a}{dt^2} = -T\sin\theta_1 + T\sin\theta_2$$

↳ + b/c positive θ_2 creates force pulling mass away from EP

~~$\Rightarrow m\frac{d^2y_a}{dt^2} = -T$~~

$$\sin\theta_1 = \frac{y_a}{\ell+\Delta\ell} \simeq \frac{y_a}{\ell}$$

$$\sin\theta_2 = \frac{(y_b-y_a)}{\ell+\Delta\ell} \simeq \frac{y_b-y_a}{\ell}$$

$$\Rightarrow m\frac{d^2y_a}{dt^2} = -\frac{T}{\ell}y_a + \frac{T}{\ell}(y_b-y_a) = \frac{T}{\ell}(y_b-2y_a)$$ [eq. 4.43]

· Eq. of motion: (mass b)

$$m\frac{d^2y_b}{dt^2} = -T\sin\theta_2 - T\sin\theta_3$$

↳ both $-$ b/c positive θ_2, θ_3 creates force toward EP

$$\sin\theta_2 \simeq \frac{y_b-y_a}{\ell}$$

$$\sin\theta_3 = \frac{y_b}{\ell}$$

$$\Rightarrow m\frac{d^2y_b}{dt^2} = (y_a-2y_b)\frac{T}{\ell}$$ [eq. 4.44]

· Use complex representation to solve

→ substitute $y_a = Ae^{i\omega t}$, $y_b = Be^{i\omega t}$

into eqs. 4.43, 4.44 above, w/ $\frac{d^2y}{dt^2} = -\omega^2 y$.

$$[4.43] \quad m\frac{d^2y_a}{dt^2} = \frac{T}{\ell}(y_b - 2y_a)$$

$$\Rightarrow -m\omega^2 y_a = \frac{T}{\ell}(y_b - 2y_a)$$

$$\Rightarrow -m\omega^2 A\cancel{e^{i\omega t}} = \frac{T}{\ell}\cancel{e^{i\omega t}}(B - 2A)$$

$$-m\omega^2 A + \frac{2AT}{\ell} = B\frac{T}{\ell} \quad \Rightarrow A\left(\frac{2T}{\ell} - m\omega^2\right) = \frac{T}{\ell}B \quad \text{[eq. 4.45]}$$

· Taking a similar approach for [4.44] gives $\frac{T}{\ell}A = B\left(\frac{2T}{\ell} - m\omega^2\right)$

· Take ratio of $\frac{A}{B}$ for each of the above and equate:

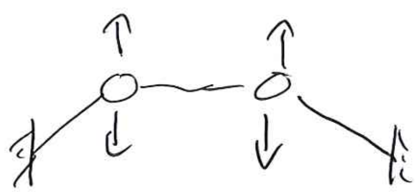
[All this is similar to what we did in lecture 9]

$$\frac{A}{B} = \left(\frac{2T}{\ell} - m\omega^2\right)^2 = \left(\frac{T}{\ell}\right)^2$$

$$\text{solutions} \Rightarrow \omega^2 = \frac{T}{m\ell} \;\; \& \;\; \frac{3T}{m\ell}$$

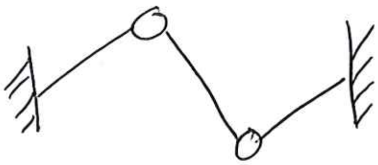
· Plug in $\omega^2 = \frac{T}{m\ell}$ into [eq 4.45] ⇒ get $A = B$

→ 1st normal mode w/ both masses oscillating in phase

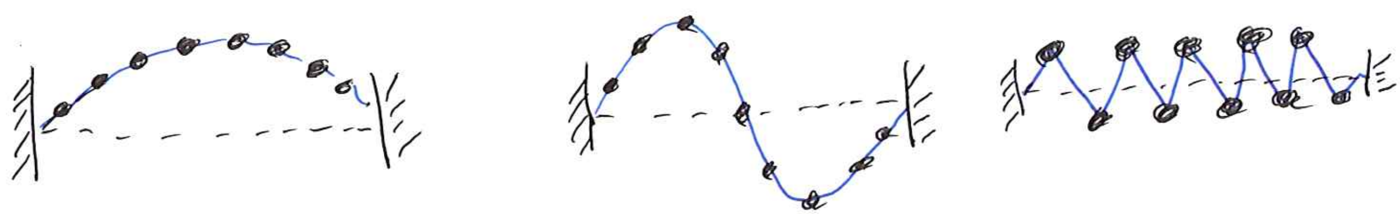


- Plug in $\omega^2 = \frac{3T}{m\ell}$ into [eq. 4.45] $\Rightarrow$ get $A = -B$

  $\rightarrow$ 2nd normal mode w/ masses moving opposite



- These two modes in this example already resemble standing <u>waves</u> on a taut string.
- This becomes more apparent when adding more masses to the problem



  $\rightarrow$ Each mass in these arrangements oscillate at <u>same</u> frequency, but with different amplitudes

- # of normal modes = # of masses (in this 1D example)
  transverse only
- Highest mode when adjacent ~~masses~~ masses move in opposite directions
  (right drawing)
- Coupled oscillators therefore bridge to <u>WAVES</u>!

$\boxed{\text{END UNIT 1}}$!